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TOPOLOGICAL DIVISOR OF ZERO PERTURBATION FUNCTIONS

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Abstract

Let $(\mathsf{A}, \|\cdot\|)$ be a complex unital Banach algebra and let $\mathcal{N}_{\mathsf{Eq}}$ be the set of all algebra-norms on A equivalent to the given algebra-norm. In this paper, we introduce the concept of $\rho_{\ell d}$- perturbation and ρ_{rd}- perturbation functions depending on a norm $\sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}$ and related to the notion of "topological divisors of zero". We prove that some usual measures of either non-compactness or non-strict-singularity of operators, as well other quantities are $\rho_{\ell d}$- perturbation or ρ_{rd}- perturbation function. We prove several spectral radius formulae for $\rho_{\ell d}$- perturbation and ρ_{rd}- perturbation functions. In particular, we prove that if $\mathcal{P}_{\sharp\cdot\sharp}$ is a $\rho_{\ell d}$- perturbation or ρ_{rd}- perturbation function and $x \in \mathsf{A}$, then

$$\rho(x) = \inf\{\mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}} : k \in \mathbb{N}^*\} = \lim_{k\to+\infty} \mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}},$$

and

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$$\rho(x) = \inf\{\mathcal{P}_{\sharp\cdot\sharp}(x) : \sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}\},$$

where $\rho(x)$ denotes the spectral radius of x.

1. Introduction

In the literature about the operator algebra, many authors ([1], [5], [10], [12], [14], [15], [18], [19], [23], [25], [26], [27], [28], [29], [30], [32], and [33],...) have introduced and investigated several quantities related to the operators, such as measures of non-compactness, measures of non-strict-singularity, numerical range, ...etc. They have established formulae for the spectral radius and the essential spectral radius of a given operator, in terms of the asymptotic behavior of its powers. The notion of "perturbation functions" has been introduced on the algebra of operators. For instance, in [10], [19], [31],..., the authors defined perturbation functions, which are invariant by compact perturbation.

In this paper, we introduce, on a given Banach algebra, new functions connected with the concept of "topological divisors of zero" and depending on a norm $\sharp\cdot\sharp$ of the algebra.

In Section 3, $\rho_{\ell d}$-perturbation and ρ_{rd}-perturbation functions are introduced. These functions will be called ρ_Z-*perturbation functions*. Note that number of well-known quantities is actually special examples of such functions (see the given examples). With these functions, we provide analogous formulae to Gelfand's spectral radius ones (see Theorems 3.4 and 3.5). For example, if $(\mathsf{A}, \|\cdot\|)$ is a complex unital Banach algebra and $\sharp\cdot\sharp$ is an algebra-norm on A equivalent to the given algebra-norm, we prove that if $\mathcal{P}_{\sharp\cdot\sharp}$ is a ρ_Z-perturbation function and $x \in \mathsf{A}$, then $\rho(x) = \inf\{\mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}} : k \in \mathbb{N}^*\} = \lim_{k\to+\infty} \mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}}$, and $\rho(x) = \inf\{\mathcal{P}_{\sharp\cdot\sharp}(x) : \sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}\}$, where $\rho(x)$ denotes the spectral radius of x.

In Section 4, A is supposed to be a C^*- algebra. Relying on Murphy-West's work, [20], we establish formula in terms of ρ_Z- perturbation function. As an example, we show that if $x \in \mathsf{A}$ and $\mathcal{P}_{\|\cdot\|}$ is a ρ_Z- perturbation function, then $\rho(x) = \inf\{\mathcal{P}_{\|\cdot\|}(e^a x e^{-a}) : a = a^* \in \mathsf{A}\}$.

2. Preliminaries

First, we introduce some notation and terminology. Let $(\mathsf{A}, \|\cdot\|)$ be a complex unital Banach algebra with unity *e*. We denote by $\mathcal{N}_{\mathsf{Eq}}(\mathsf{A})$ the set of all algebra-norms on A equivalent to the given algebra-norm and satisfying $\sharp e \sharp = 1$ for all $\sharp \cdot \sharp \in \mathcal{N}_{\mathsf{Eq}}(\mathsf{A})$. When no confusion can arise, we simply write $\mathcal{N}_{\mathsf{Eq}}$.

Let $x, y \in \mathsf{A}$ such that $xy = e$, then *x* is called a *left inverse of y* and *y* is said to be a *right inverse of x*. If an element *x* is both a left inverse and a right inverse of *y,* then *x* is called a *two-sided inverse*, or *simply an inverse of y*. An element *x* with an inverse in A is said to be *invertible in* A. Remark that this inverse is unique and will be denoted by x^{-1}. The set of all invertible elements of A is denoted by $\mathsf{Inv}(\mathsf{A})$. An element with a left (resp., right) inverse is left (resp., right) invertible. The set of all left (resp., right) invertible elements of A is denoted by $\mathsf{Inv}_{\ell}(\mathsf{A})$ (resp., $\mathsf{Inv}_{r}(\mathsf{A})$).

An element $x \in \mathsf{A}$ is called a *left topological divisor of zero*, if $\inf\{\sharp xa \sharp : a \in \mathsf{A}, \sharp a \sharp = 1\} = 0$. Similarly, *x* is a *right topological divisor of zero*, if $\inf\{\sharp ax \sharp : a \in \mathsf{A}, \sharp a \sharp = 1\} = 0$. We will denote by

$$\mathsf{Inv}_{\ell d}(\mathsf{A}) = \{x \in \mathsf{A} : x \text{ is not left topological divisor of zero}\}.$$

The set $\mathsf{Inv}_{rd}(\mathsf{A})$ is defined in the obvious way.

The spectrum of $x \in \mathsf{A}$ is denoted by $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda e \notin \mathsf{Inv}(\mathsf{A})\}$ and its spectral radius $\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$. The *left approximate point spectrum* of *x,* denoted by $\sigma_{\ell d}(x)$, is the set

$$\sigma_{\ell d}(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is a left topological divisor of zero}\};$$

and the *right approximate point spectrum* of *x*, denoted by $\sigma_{rd}(x)$, is the set

$$\sigma_{rd}(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is a right topological divisor of zero}\}.$$

It is well-known that

$$\mathsf{Inv}_{\ell}(\mathsf{A}) \subseteq \mathsf{Inv}_{\ell d}(\mathsf{A}) \quad \text{and} \quad \mathsf{Inv}_{r}(\mathsf{A}) \subseteq \mathsf{Inv}_{rd}(\mathsf{A}), \tag{2.1}$$

$$\sigma_{\ell d}(x) \cup \sigma_{rd}(x) \subseteq \sigma(x), \ \forall x \in \mathsf{A}, \tag{2.2}$$

$$\partial\sigma(x) \subseteq \sigma_{\ell d}(x) \cap \sigma_{rd}(x), \ \forall x \in \mathsf{A}, \tag{2.3}$$

where $\partial\sigma(x)$ denotes the boundary of $\sigma(x)$.

3. ρ_Z- Perturbation Functions

In this section, we introduce two perturbation functions connected with the concept of left and right topological divisors of zero and depending on the norm $\sharp \cdot \sharp$ of the algebra A. With these functions, we provide analogous formulae to Gelfand's spectral radius ones.

Recall that a function $\mathcal{S} : \mathsf{A} \to \mathbb{C}$ is absolutely homogeneous, if

$$\mathcal{S}(\lambda x) = |\lambda|\mathcal{S}(x), \ \forall \lambda \in \mathbb{C}, \ \forall x \in \mathsf{A}.$$

Definition 3.1. Given $\sharp \cdot \sharp \in \mathcal{N}_{\mathsf{Eq}}$, a $\rho_{\ell d}$- perturbation function on A is a non-negative absolutely homogeneous function $\mathcal{P}_{\sharp \cdot \sharp}$ with the following properties:

(1) $\mathcal{P}_{\sharp \cdot \sharp}(x) \leq \sharp x \sharp, \ \forall x \in \mathsf{A}$;

(2) if $x \in \mathsf{A}$ such that $\mathcal{P}_{\sharp \cdot \sharp}(x) < 1$, then $x - e \in \mathsf{Inv}_{\ell d}(\mathsf{A})$.

The ρ_{rd}-perturbation function is defined in the obvious way.

A function is said to be a ρ_Z-*perturbation*, if it is a $\rho_{\ell d}$-perturbation or ρ_{rd}-perturbation function.

It is easy to see that if $\mathcal{P}_{\sharp\cdot\sharp}$ is a ρ_Z-perturbation function, then $\mathcal{P}_{\sharp\cdot\sharp}(e) = 1$.

From (2.3), it is not difficult to see the following.

Proposition 3.2. *Let* $\sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}$ *and* $\Gamma_{\sharp\cdot\sharp} : \mathsf{A} \to \mathbb{R}_+$ *be an absolutely homogeneous function such that*

$$\rho(x) \le \Gamma_{\sharp\cdot\sharp}(x) \le \sharp x \sharp, \ \forall x \in \mathsf{A},$$

then $\Gamma_{\sharp\cdot\sharp}$ *is a* $\rho_{\ell d}$-*perturbation and a* ρ_{rd}-*perturbation function.*

Example 3.1. For every $x \in \mathsf{A}$, define:

$$\theta^{\ell}_{\sharp\cdot\sharp}(x) = \inf\{\sharp ax\sharp : a \in \mathsf{Inv}(\mathsf{A}), \sharp a\sharp = \sharp a^{-1}\sharp = 1\},$$

$$\theta^{r}_{\sharp\cdot\sharp}(x) = \inf\{\sharp xa\sharp : a \in \mathsf{Inv}(\mathsf{A}), \sharp a\sharp = \sharp a^{-1}\sharp = 1\}.$$

It is not difficult to show that the function $\theta^{\ell}_{\sharp\cdot\sharp}$ (resp., $\theta^{r}_{\sharp\cdot\sharp}$) is a $\rho_{\ell d}$-perturbation and a ρ_{rd}-perturbation function. □

Let us consider the following sets:

$$\mathsf{S}_{\sharp\cdot\sharp}(\mathsf{A}) = \{x \in \mathsf{A} : \sharp x\sharp = 1\},$$

$$\mathsf{S}^{\ell}_{\sharp\cdot\sharp}(\mathsf{A}) = \{x \in \mathsf{Inv}_{\ell}(\mathsf{A}) \cap \mathsf{S}_{\sharp\cdot\sharp}(\mathsf{A}) : \exists y \in \mathsf{S}_{\sharp\cdot\sharp}(\mathsf{A}) | yx = e\},$$

$$\mathsf{S}^{r}_{\sharp\cdot\sharp}(\mathsf{A}) = \{x \in \mathsf{Inv}_{r}(\mathsf{A}) \cap \mathsf{S}_{\sharp\cdot\sharp}(\mathsf{A}) : \exists y \in \mathsf{S}_{\sharp\cdot\sharp}(\mathsf{A}) | xy = e\}.$$

Example 3.2. For every $x \in \mathsf{A}$, define:

$$\omega^{r}_{\sharp\cdot\sharp} : x \mapsto \omega^{r}_{\sharp\cdot\sharp}(x) = \inf\{\sharp xa\sharp : a \in \mathsf{S}^{r}_{\sharp\cdot\sharp}(\mathsf{A}) \setminus \mathsf{Inv}(\mathsf{A})\},$$

$$\omega^{\ell}_{\sharp\cdot\sharp} : x \mapsto \omega^{\ell}_{\sharp\cdot\sharp}(x) = \inf\{\sharp ax\sharp : a \in \mathsf{S}^{\ell}_{\sharp\cdot\sharp}(\mathsf{A}) \setminus \mathsf{Inv}(\mathsf{A})\}.$$

$\omega^{r}_{\sharp\cdot\sharp}$ is a ρ_{rd}-perturbation function. Let $x \in \mathsf{A}$ such that $\omega^{r}_{\sharp\cdot\sharp}(x) < 1$. We know that there exists $a_0 \in \mathsf{S}^{r}_{\sharp\cdot\sharp}(\mathsf{A}) \setminus \mathsf{Inv}(\mathsf{A})$ such that $\sharp xa_0\sharp < 1$. Let $b_0 \in \mathsf{A}$ be a right inverse of a_0 such that $\sharp b_0\sharp = 1$. We have

$$\sharp x\sharp \leq \sharp xa_0b_0\sharp \leq \sharp xa_0\sharp\,\sharp b_0\sharp < \sharp b_0\sharp.$$

So, $\sharp xa_0\sharp \leq \sharp x\sharp < \sharp b_0\sharp$. Hence, by [22, Theorem 1.4.6], we get

$$(x - e)a_0 = xa_0 - a_0 \in \mathsf{Inv}_r(\mathsf{A}).$$

Consequently, $x - e \in \mathsf{Inv}_{rd}(\mathsf{A})$.

In the same way, we prove that $\omega^{\ell}_{\sharp\cdot\sharp}$ is a $\rho_{\ell d}$-perturbation function.

□

Example 3.3. Let us define

$$\tau^{\ell}_{\sharp\cdot\sharp} : x \mapsto \tau^{\ell}_{\sharp\cdot\sharp}(x) = \sup\{\rho(ax) : a \in \mathsf{S}_{\sharp\cdot\sharp}(\mathsf{A})\},$$

$$\tau^{r}_{\sharp\cdot\sharp} : x \mapsto \tau^{r}_{\sharp\cdot\sharp}(x) = \sup\{\rho(xa) : a \in \mathsf{S}_{\sharp\cdot\sharp}(\mathsf{A})\},$$

$$\tau^{\ell r}_{\sharp\cdot\sharp} : x \mapsto \tau^{\ell r}_{\sharp\cdot\sharp}(x) = \sup\{\rho(axb) : a, b \in \mathsf{A}, \sharp a\sharp\,\sharp b\sharp = 1\}.$$

It is not difficult to see that $\tau^{\ell}_{\sharp\cdot\sharp}$ (resp., $\tau^{r}_{\sharp\cdot\sharp}$, $\tau^{\ell r}_{\sharp\cdot\sharp}$) is a $\rho_{\ell d}$-perturbation and a ρ_{rd}-perturbation function. □

Let A' denote the dual space of A, i.e., the Banach space of all continuous linear functionals on A. For $x \in \mathsf{A}$, the *numerical range* of x, denoted $\mathcal{V}_{\sharp\cdot\sharp}(x)$, is defined by

$$\mathcal{V}_{\sharp\cdot\sharp}(x) = \{f(x) : f \in \mathsf{A}', f(e) = 1 = \sharp f\sharp\},$$

and its *numerical radius* denoted by $\mathsf{v}_{\sharp\cdot\sharp}(x)$, is defined by

$$\mathsf{v}_{\sharp\cdot\sharp}(x) = \sup\{|\lambda| : \lambda \in \mathcal{V}(x)\}.$$

From ([24], [4, Proposition 6, p. 53] or [3, Theorem 6, p. 19]), we know that

$$\rho(x) \leq \mathsf{v}_{\sharp\cdot\sharp}(x) \leq \sharp x\sharp,\ \forall x \in \mathsf{A}.$$

Example 3.4. By Proposition 3.2, we deduce that the function $\mathsf{v}_{\sharp\cdot\sharp} : \mathsf{A} \to \mathbb{R}_+$ defined by $x \mapsto \mathsf{v}_{\sharp\cdot\sharp}(x)$ is a $\rho_{\ell d}$-perturbation and a ρ_{rd}-perturbation. □

Example 3.5. For $x \in \mathsf{A}$, let

$$\mathsf{v}^{\ell}_{\sharp\cdot\sharp}(x) = \sup\{\mathsf{v}_{\sharp\cdot\sharp}(ax) : a \in \mathsf{A},\ \sharp a\sharp = 1\},$$

$$\mathsf{v}^{r}_{\sharp\cdot\sharp}(x) = \sup\{\mathsf{v}_{\sharp\cdot\sharp}(xa) : a \in \mathsf{A},\ \sharp a\sharp = 1\},$$

$$\mathsf{v}^{\ell r}_{\sharp\cdot\sharp}(x) = \sup\{\mathsf{v}_{\sharp\cdot\sharp}(axb) : a,\ b \in \mathsf{A},\ \sharp a\sharp\,\sharp b\sharp = 1\}.$$

We can prove without any difficulty that the functions $\mathsf{v}^{\ell}_{\sharp\cdot\sharp}$, $\mathsf{v}^{r}_{\sharp\cdot\sharp}$, and $\mathsf{v}^{\ell r}_{\sharp\cdot\sharp}$ are $\rho_{\ell d}$-perturbation and ρ_{rd}-perturbation functions. □

Example 3.6. For $x \in \mathsf{A}$, we define the function $\overline{\mathsf{v}}_{\sharp\cdot\sharp}$ by

$$\overline{\mathsf{v}}_{\sharp\cdot\sharp}(x) = \inf\{\mathsf{v}_{\sharp\cdot\sharp}(axa^{-1}) : a \in \mathsf{Inv}(\mathsf{A}),\ \sharp a\sharp\,\sharp a^{-1}\sharp = 1\}.$$

It is clear that the function $\overline{\mathsf{v}}_{\sharp\cdot\sharp}$ is a $\rho_{\ell d}$-perturbation and a ρ_{rd}-perturbation functions. □

Proposition 3.3. *Let $\sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}$ and $\Gamma_{\sharp\cdot\sharp} : \mathsf{A} \to \mathbb{R}_+$ be an absolutely homogeneous function such that*

$$\rho(x) \leq \Gamma_{\sharp\cdot\sharp}(x) \leq \sharp x\sharp, \forall x \in \mathsf{A},$$

then the function $\hat{\Gamma}_{\sharp\cdot\sharp} : \mathsf{A} \to \mathbb{R}_+$, defined by

$$\hat{\Gamma}_{\sharp\cdot\sharp}(x) = \inf\{\Gamma_{\sharp\cdot\sharp}(bxa) : a \in \mathsf{Inv}_r(\mathsf{A}) \setminus \mathsf{Inv}(\mathsf{A}),\ b \in \mathsf{A},\ ab = e,\ \sharp a\sharp\,\sharp b\sharp = 1\},$$

is a $\rho_{\ell d}$-perturbation and a ρ_{rd}-perturbation functions.

Proof. To verify property (2), let $x \in \mathsf{A}$ such that $\widehat{\Gamma}_{\sharp\cdot\sharp}(x) < 1$. From the definition of $\widehat{\Gamma}_{\sharp\cdot\sharp}$, we know that there exist $a_0 \in \mathsf{Inv}_{r}(\mathsf{A}) \setminus \mathsf{Inv}(\mathsf{A})$ and $b_0 \in \mathsf{A}$ such that $\Gamma_{\sharp\cdot\sharp}(b_0 x a_0) < 1$, $a_0 b_0 = e$, and $\sharp a_0 \sharp \sharp b_0 \sharp = 1$. So, from [2, Corollary 1, p. 2] (or [16, Proposition 3.2.8]), we deduce

$$\rho(x) = \rho(x a_0 b_0) = \rho(b_0 x a_0) \leq \Gamma_{\sharp\cdot\sharp}(b_0 x a_0) < 1.$$

This implies $x - e \in \mathsf{Inv}(\mathsf{A})$. □

Example 3.7. For every $x \in \mathsf{A}$, define

$$\widehat{\sharp\cdot\sharp}(x) = \inf\{\sharp bxa \sharp : a \in \mathsf{Inv}_{r}(\mathsf{A}) \setminus \mathsf{Inv}(\mathsf{A}),\ b \in \mathsf{A},\ ab = e,\ \sharp a \sharp \sharp b \sharp = 1\},$$

$$\widehat{\mathsf{v}}_{\sharp\cdot\sharp}(x) = \inf\{\mathsf{v}(bxa) : a \in \mathsf{Inv}_{r}(\mathsf{A}) \setminus \mathsf{Inv}(\mathsf{A}),\ b \in \mathsf{A},\ ab = e,\ \sharp a \sharp \sharp b \sharp = 1\}.$$

It follows from Proposition 3.3, that $\widehat{\sharp\cdot\sharp}$ (resp., $\widehat{\mathsf{v}}_{\sharp\cdot\sharp}$) is a $\rho_{\ell d}$- perturbation and a ρ_{rd}- perturbation functions.

For the convenience of the reader, some notions and definitions in operators theory will be recalled here.

Throughout, $(\mathsf{X}, \|\cdot\|)$ denotes a Banach space and $(\mathcal{B}(\mathsf{X}), \|\cdot\|)$ the algebra of bounded operators from X into itself. Let $T \in \mathcal{B}(\mathsf{X})$ and $\sharp\cdot\sharp$ a norm on X equivalent to $\|\cdot\|$, we will denote $\sharp T \sharp$ as the algebra-norm for T relative to the $\sharp\cdot\sharp$ norm. The conjugate space of the Banach space X is denoted by X^* and the adjoint of a linear operator T in $\mathcal{B}(\mathsf{X})$ by T^*. In case where X is a Hilbert space, we denote it by H.

For an operator $T \in \mathcal{B}(\mathsf{X})$, we denote, respectively, by $\mathsf{N}(T)$ and $\mathsf{R}(T)$, the kernel and the range of T. The identity operator will be denoted by I.

Let $\mathcal{K}(\mathsf{X})$ be the ideal of compact operators. We shall use π to denote the natural homomorphism of $\mathcal{B}(\mathsf{X})$ onto the *Calkin algebra* $\mathcal{C}(\mathsf{X}) = \mathcal{B}(\mathsf{X}) / \mathcal{K}(\mathsf{X})$.

An operator $T \in \mathcal{B}(\mathsf{X})$ is called *Fredholm* (resp., *semi-Fredholm*), if $\mathsf{R}(T)$ is closed subspace of X and $\max\{\dim \mathsf{N}(T), \dim \mathsf{N}(T^*)\} < \infty$ (resp., $\min\{\dim \mathsf{N}(T), \dim \mathsf{N}(T^*)\} < \infty$). It is well known (Atkinson's theorem) that T is a Fredholm, if and only if $\pi(T)$ is invertible in $\mathcal{C}(\mathsf{X})$. For semi-Fredholm theory and Calkin algebras, see, for instance, [6-8, 17]. If T is a semi-Fredholm, the index of T is defined by

$$\operatorname{ind}(T) = \dim \mathsf{N}(T) - \dim \mathsf{N}(T^*).$$

If $T \in \mathcal{B}(\mathsf{X})$, then $\sigma_e(T) = \sigma(\pi(T))$ denotes the essential spectrum of T, while $\sigma_e^{\pm}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not semi-Fredholm$\}$ denotes the semi-Fredholm spectrum of T.

For $T \in \mathcal{B}(\mathsf{X})$ and M a subspace of X, we denote by $T_{|\mathsf{M}}$ the restriction of T to M. An operator $T \in \mathcal{B}(\mathsf{X})$ is *strictly singular*, if for every infinite dimensional closed subspace M of X, $T_{|\mathsf{M}}$ is not a homomorphism. For more details about strictly singular operator, see [6, 13]. It is well known that if K is a compact operator, then K is strictly singular operator.

Various different measures of non-compactness of bounded operators have appeared in the literature. In the following, we show that the usual measures of non-compactness and other related quantities are ρ_Z-perturbation functions.

Example 3.8. For $T \in \mathcal{B}(\mathsf{X})$, we denote by

$$\widehat{\sigma_e}(T) = \sigma_e(T) \cup \overline{\{\mu : \{\mu\} \text{ is a component of } \sigma(T)\}}.$$

The function $\widehat{\rho_e} : \mathcal{B}(\mathsf{X}) \to \mathbb{R}_+$ defined by

$$\widehat{\rho_e}(T) = \sup\{\lambda : \lambda \in \widehat{\sigma_e}(T)\},$$

is a $\rho_{\ell d}$- perturbation and a ρ_{rd}- perturbation functions. We prove first that $\partial\sigma(T) \subseteq \widehat{\sigma_e}(T)$. Let $\lambda \in \partial\sigma(T)$, assume that $\lambda \notin \sigma_e(T)$. So $T - \lambda I$ is Fredholm. Now, by [13, Corollary V.1.7] and [13, Theorem V.1.6], we know that there exists $\varepsilon > 0$ such that $T - \mu I$ is Fredholm and $\dim \mathsf{N}(T - \mu I)$ and $\dim \mathsf{N}((T - \mu I)^*)$ are constant for all $\mu \in \mathsf{D}(\lambda, \varepsilon) \setminus \{\lambda\}$, where $\mathsf{D}(\lambda, \varepsilon)$ denotes the open disk centered at λ with radius ε.

Since $\lambda \in \partial\sigma(T)$, we deduce that

$$\dim \mathsf{N}(T - \mu I) = \dim \mathsf{N}((T - \mu I)^*) = 0, \ \forall \mu \in \mathsf{D}(\lambda, \varepsilon) \setminus \{\lambda\}.$$

This implies that the singleton set $\{\lambda\}$ is a component of $\sigma(T)$.

Now, if $\widehat{\rho_e}(T) < 1$, we deduce that $\rho(T) < 1$. So, $T - I$ is an invertible operator. □

Example 3.9. Similarly as in Example 3.8, we prove that

$$\widehat{\sigma_e}^{\pm}(T) = \sigma_e^{\pm}(T) \cup \overline{\{\mu : \{\mu\} \text{ is a component of } \sigma(T)\}},$$

is a $\rho_{\ell d}$- perturbation and a ρ_{rd}- perturbation functions. □

Example 3.10. If Ω is a non-empty bounded subset of X, then Kuratowski measure of non-compactness of Ω, denoted $\psi_{\sharp\cdot\sharp}(\Omega)$, is given by

$$\psi_{\sharp\cdot\sharp}(\Omega) = \inf\{r > 0 : \Omega \text{ can be covered by finitely many sets of diameter } \leq r\}.$$

For $T \in \mathcal{B}(\mathsf{X})$ the Kuratowski measure of non-compactness of T is denoted by $\Psi_{\sharp\cdot\sharp}(T)$, and defined by

$$\Psi_{\sharp\cdot\sharp}(T) = \sup\left\{\frac{\psi_{\sharp\cdot\sharp}(T(\Omega))}{\psi_{\sharp\cdot\sharp}(\Omega)} : \psi_{\sharp\cdot\sharp}(\Omega) > 0\right\}.$$

From [9, Lemma I.2.8], we know that $\Psi_{\sharp\cdot\sharp}(T) = \Psi_{\sharp\cdot\sharp}(T + K)$ for all $K \in \mathcal{K}(\mathsf{X})$, thus $\Psi_{\sharp\cdot\sharp}$ induces a map $\widetilde{\Psi}_{\sharp\cdot\sharp} : \mathcal{C}(\mathsf{X}) \to \mathbb{R}_+$ such that $\widetilde{\Psi}_{\sharp\cdot\sharp} \circ \pi = \Psi_{\sharp\cdot\sharp}$. From [9, Theorem I.4.4], it is not difficult to verify that $\widetilde{\Psi}_{\sharp\cdot\sharp}$ is a $\rho_{\ell d}$- perturbation and a ρ_{rd}- perturbation functions. □

Example 3.11. The measure of non-compactness of Housdorff. We define the function $\tilde{\beta}_{\sharp\cdot\sharp} : \mathcal{B}(\mathsf{X}) \to \mathbb{R}_+$ by

$$\tilde{\beta}_{\sharp\cdot\sharp}(T) = \inf\left\{ r > 0 : T(\mathsf{B}_{\sharp\cdot\sharp}(0, 1)) \subseteq \bigcup_{i=1}^{n} \mathsf{B}_{\sharp\cdot\sharp}(x_i, r) \right\},$$

where $\mathsf{B}_{\sharp\cdot\sharp}(x_i, r) = \{x \in \mathsf{X} : \sharp x - x_i \sharp < r\}$. From [9, Lemma I.2.8], we know that

$$\tilde{\beta}_{\sharp\cdot\sharp}(T) = \tilde{\beta}_{\sharp\cdot\sharp}(T + K),\ \forall K \in \mathcal{K}(\mathsf{X}),$$

thus $\tilde{\beta}_{\sharp\cdot\sharp}$ induces a map $\Phi_{\sharp\cdot\sharp} : \mathcal{C}(\mathsf{X}) \to \mathbb{R}_+$ such that $\Phi_{\sharp\cdot\sharp} \circ \pi = \tilde{\beta}_{\sharp\cdot\sharp}$. From [9, Theorem I.4.4], it is not difficult to see that $\Phi_{\sharp\cdot\sharp}$ is a $\rho_{\ell d}$- perturbation and a ρ_{rd}- perturbation functions. □

Example 3.12. Measure of non-compactness (see [18] and [9, p. 24]). For $T \in \mathcal{B}(\mathsf{X})$, let

$$\sharp\cdot\sharp_m : T \mapsto \sharp T \sharp_m = \inf\{\, \sharp T_{|\mathsf{M}} \sharp : \operatorname{codim} \mathsf{M} < \infty \,\}.$$

By [9, Corollary I.2.22] (see also [18, p. 9]), we know that

$$\sharp T \sharp_m = \sharp T + K \sharp_m,\ \forall K \in \mathcal{K}(\mathsf{X}).$$

Thus $\sharp\cdot\sharp_m$ induces a map $\Theta_m : \mathcal{C}(\mathsf{X}) \to \mathbb{R}_+$ such that $\Theta_m \circ \pi(T) = \sharp T \sharp_m$ for all $T \in \mathcal{B}(\mathsf{X})$. The function Θ_m is a $\rho_{\ell d}$- perturbation and a ρ_{rd}- perturbation functions. Clearly, Θ_m has property (1). The property (2) follows from [18, Theorem 6.1]. □

Example 3.13. Measures of non-strict singularity (see [14, 23, 26, 29]). For $T \in \mathcal{B}(\mathsf{H})$, we make the following definitions:

$$G_{\sharp\cdot\sharp}^{\mathsf{M}}(T) = \inf\{\, \sharp T_{|\mathsf{N}}\sharp : \mathsf{N} \text{ subspace of } \mathsf{M},\ \dim \mathsf{N} = +\infty \,\},$$

$$\Delta_{\sharp\cdot\sharp}(T) = \sup\{\, G_{\sharp\cdot\sharp}^{\mathsf{M}}(T) : \mathsf{M} \text{ subspace of } \mathsf{H},\ \dim \mathsf{M} = +\infty \,\}.$$

In [23, p. 1062], we know that

$$\Delta_{\sharp\cdot\sharp}(K) = 0 \Leftrightarrow K \text{ is strictly singular}$$

$$\Leftrightarrow \Delta_{\sharp\cdot\sharp}(T + K) = \Delta_{\sharp\cdot\sharp}(T),\ \forall T \in \mathcal{B}(\mathsf{H}).$$

Thus $\Delta_{\sharp\cdot\sharp}$ induces a map $\tilde{\Delta}_{\sharp\cdot\sharp} : \mathcal{C}(\mathsf{H}) \to \mathbb{R}_+$ such that $\tilde{\Delta}_{\sharp\cdot\sharp} \circ \pi = \Delta_{\sharp\cdot\sharp}$. The function $\tilde{\Delta}_{\sharp\cdot\sharp}$ is a $\rho_{\ell d}$-perturbation since property (2) is a consequence of [23, Theorem 2.12]. □

Example 3.14. Measures of non-strict singularity (see [11, 23, 30]). For $T \in \mathcal{B}(\mathsf{X})$, let

$$\tau_{\sharp\cdot\sharp}(T) = \sup\{ m(T_{|\mathsf{M}}) : \mathsf{M} \text{ subspace of } \mathsf{X},\ \dim \mathsf{M} = +\infty \},$$

where $m(T_{|\mathsf{M}}) = \inf\{\sharp T(x)\sharp : x \in \mathsf{M},\ \sharp x \sharp = 1\}$. In [23, p. 1062], we know that

$$\tau_{\sharp\cdot\sharp}(K) = 0 \Leftrightarrow K \text{ is strictly singular}$$

$$\Leftrightarrow \tau_{\sharp\cdot\sharp}(T + K) = \tau_{\sharp\cdot\sharp}(T),\ \forall T \in \mathcal{B}(\mathsf{X}).$$

Thus $\tau_{\sharp\cdot\sharp}$ induces a map $\tilde{\tau}_{\sharp\cdot\sharp} : \mathcal{C}(\mathsf{X}) \to \mathbb{R}_+$ such that $\tilde{\tau}_{\sharp\cdot\sharp} \circ \pi = \tau_{\sharp\cdot\sharp}$.

$\tilde{\tau}_{\sharp\cdot\sharp}$ is a $\rho_{\ell d}$-perturbation and a ρ_{rd}-perturbation functions. Indeed, property (1) in Definition 3.1 is obvious. The property (2) is a consequence of [23, Corollary 2.15]. □

We prove for a $\rho_{\ell d}$-perturbation function an analogue of Gelfand's spectral radius formula.

Theorem 3.4. *Let* $x \in \mathsf{A}$, $\sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}$, *and let* $\mathcal{P}_{\sharp\cdot\sharp}$ *be a* $\rho_{\ell d}$-*perturbation function. Then*

$$\rho(x) = \inf\{\mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}} : k \in \mathbb{N}^*\}$$

$$= \lim_{k\to+\infty} \mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}}.$$

Proof. We first prove that

$$\rho(x) \leq \inf\{\mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}} : k \in \mathbb{N}^*\}.$$

Let $k \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}$, such that $|\lambda| > \mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}}$. From Definition 3.1, we have

$$\mathcal{P}_{\sharp\cdot\sharp}\left(\frac{1}{\lambda^k} x^k\right) \leq \frac{1}{|\lambda|^k} \mathcal{P}_{\sharp\cdot\sharp}(x^k) < 1.$$

So, $x^k - \lambda^k e \in \mathsf{Inv}_{\ell d}(\mathsf{A})$. Since $\sigma_{\ell d}(x^k) = \sigma_{\ell d}(x)^k$, we deduce that

$$\sup\{|\lambda| \in \mathbb{C} : \lambda \in \sigma_{\ell d}(x)\} \leq \mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}}.$$

By (2.3), we get

$$\rho(x) \leq \mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}}.$$

This implies

$$\rho(x) \leq \inf\{\mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}} : k \in \mathbb{N}^*\}. \tag{1}$$

On the other hand, by Definition 3.1, for any $k \geq 1$, we have

$$\mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}} \leq \sharp x^k \sharp^{\frac{1}{k}}.$$

So,

$$\overline{\lim_{k\to+\infty}} \mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}} \leq \overline{\lim_{k\to+\infty}} \sharp x^k \sharp^{\frac{1}{k}}. \tag{2}$$

Since, $\sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}$, we deduce that

$$\overline{\lim_{k\to+\infty}} \sharp x^k \sharp^{\frac{1}{k}} = \overline{\lim_{k\to+\infty}} \|x^k\|^{\frac{1}{k}} = \rho(x). \tag{3}$$

Now, (2) and (3) imply that

$$\overline{\lim_{k\to+\infty}}\, \mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}} \leq \rho(x).$$

Now, the result is an immediate consequence of (1) and (3). This completes the proof. □

Remark. Let us remark that Theorem 6.3 in [18] follows from Theorem 3.4 (see, Example 3.12).

Similarly, one can prove the following:

Theorem 3.5. *Let* $x \in \mathsf{A}$, $\sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}$, *and let* $\mathcal{P}_{\sharp\cdot\sharp}$ *be a* ρ_{rd}-*perturbation function. Then*

$$\rho(x) = \inf\{\mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}} : k \in \mathbb{N}^*\}$$

$$= \lim_{k\to+\infty} \mathcal{P}_{\sharp\cdot\sharp}(x^k)^{\frac{1}{k}}.$$

For the convenience of the reader, the following result from [3, Lemma 8, p. 21] is stated.

Lemma 3.6. *Let* $x \in \mathsf{A}$ *and* $\varepsilon > 0$. *Then, there exists* $\sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}$ *such that* $\sharp x\sharp \leq \rho(x) + \varepsilon$.

Theorem 3.7. *Let* $\mathcal{P}_{\sharp\cdot\sharp}$ *be a* $\rho_{\ell d}$-*perturbation function. Then*

$$\rho(x) = \inf\{\mathcal{P}_{\sharp\cdot\sharp}(x) : \sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}\},\ \forall x \in \mathsf{A}.$$

Proof. Let $\sharp\cdot\sharp \in \mathcal{N}_{\mathsf{Eq}}$ and $\lambda \in \mathbb{C}$, such that $|\lambda| > \mathcal{P}_{\sharp\cdot\sharp}(x)$. From Definition 3.1, we have

$$\mathcal{P}_{\sharp\cdot\sharp}\left(\frac{1}{\lambda}x\right) \leq \frac{1}{|\lambda|}\mathcal{P}_{\sharp\cdot\sharp}(x) < 1.$$

This implies $x - \lambda e \in \mathsf{Inv}_{\ell d}(\mathsf{A})$, and so

$$\sup\{|\lambda| : \lambda \in \sigma_{\ell d}(x)\} \leq \mathcal{P}_{\sharp\cdot\sharp}(x).$$

Now by (2.3), we obtain

$$\rho(x) \leq \inf \{\mathcal{P}_{\sharp\cdot\sharp}(x) : \sharp \cdot \sharp \in \mathcal{N}_{\mathsf{Eq}}\}. \tag{1}$$

For the converse, let $\varepsilon > 0$. By Lemma 3.6, we know that there exists $\sharp \cdot \sharp_{\varepsilon} \in \mathcal{N}_{\mathsf{Eq}}$ such that $\sharp x \sharp_{\varepsilon} \leq \rho(x) + \varepsilon$.

This proves that

$$\mathcal{P}_{\sharp\cdot\sharp_{\varepsilon}}(x) \leq \sharp x \sharp_{\varepsilon} \leq \rho(x) + \varepsilon.$$

Therefore,

$$\inf \{\mathcal{P}_{\sharp\cdot\sharp}(x) : \sharp \cdot \sharp \in \mathcal{N}_{\mathsf{Eq}}\} \leq \rho(x) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$\inf \{\mathcal{P}_{\sharp\cdot\sharp}(x) : \sharp \cdot \sharp \in \mathcal{N}_{\mathsf{Eq}}\} \leq \rho(x). \tag{2}$$

Now, by (1) and (2), we get

$$\inf \{\mathcal{P}_{\sharp\cdot\sharp}(x) : \sharp \cdot \sharp \in \mathcal{N}_{\mathsf{Eq}}\} = \rho(x).$$

This completes the proof. □

We can obtain the following theorem in the same way as Theorem 3.7.

Theorem 3.8. *Let* $\mathcal{P}_{\sharp\cdot\sharp}$ *be a* ρ_{rd}-*perturbation function. Then*

$$\rho(x) = \inf \{\mathcal{P}_{\sharp\cdot\sharp}(x) : \sharp \cdot \sharp \in \mathcal{N}_{\mathsf{Eq}}\}, \ \forall x \in \mathsf{A}.$$

4. Case of C^*-Algebras

We conclude this paper with other results on ρ_Z-perturbation functions in the case of C^*-algebras.

Throughout this section, we will assume that A is a unital C^*-algebra.

Theorem 4.1. *Let* $x \in \mathsf{A}$ *and let* $\mathcal{P}_{\|\cdot\|}$ *be a* $\rho_{\ell d}$*-perturbation function. Then*

$$\rho(x) = \inf \{\mathcal{P}_{\|\cdot\|}(e^a x e^{-a}) : a = a^* \in \mathsf{A}\}$$

$$= \inf \{\mathcal{P}_{\|\cdot\|}(bxb^{-1}) : b \in \mathsf{Inv}(\mathsf{A})\}.$$

Proof. From the proof of Theorem 3.7, we see that $\rho(x) \leq \mathcal{P}_{\|\cdot\|}(x)$. Since $\rho(bxb^{-1}) = \rho(x)$ for all $b \in \mathsf{Inv}(\mathsf{A})$, we deduce that

$$\rho(x) \leq \inf \{\mathcal{P}_{\|\cdot\|}(bxb^{-1}) : b \in \mathsf{Inv}(\mathsf{A})\}$$

$$\leq \inf \{\mathcal{P}_{\|\cdot\|}(e^a x e^{-a}) : a = a^* \in \mathsf{A}\}.$$

Since $\mathcal{P}_{\|\cdot\|}(x) \leq \|x\|$, we conclude from [20, Proposition 4], that

$$\rho(x) = \inf \{\|e^a x e^{-a}\| : a = a^* \in \mathsf{A}\}$$

$$\geq \inf\{\mathcal{P}_{\|\cdot\|}(e^a x e^{-a}) : a = a^* \in \mathsf{A}\}.$$

This completes the proof of the theorem. □

We can obtain the following theorem in the same way as Theorem 4.1.

Theorem 4.2. *Let* $x \in \mathsf{A}$ *and let* $\mathcal{P}_{\|\cdot\|}$ *be a* ρ_{rd}*-perturbation function. Then*

$$\rho(x) = \inf \{\mathcal{P}_{\|\cdot\|}(e^a x e^{-a}) : a = a^* \in \mathsf{A}\}$$

$$= \inf \{\mathcal{P}_{\|\cdot\|}(bxb^{-1}) : b \in \mathsf{Inv}(\mathsf{A})\}.$$

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